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The dS/CFT correspondence

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Abstract: A holographic duality is proposed relating quantum gravity on dS_D (D-dimensional de Sitter space) to conformal field theory on a single S^{D-1} ((D-1)-sphere), in which bulk de Sitter correlators with points on the boundary are related to CFT correlators on the sphere, and points on $\mathcal{I}^+$ (the future boundary of dS_D) are mapped to the antipodal points on S^{D-1} relative to those on $\mathcal{I}^-$. For the case of dS_3, which is analyzed in some detail, the central charge of the CFT_2 is computed in an analysis of the asymptotic symmetry group at $\mathcal{I}^\pm$. This dS/CFT proposal is supported by the computation of correlation functions of a massive scalar field. In general the dual CFT may be non-unitary and (if for example there are sufficently massive stable scalars) contain complex conformal weights. We also consider the physical region $\mathcal{O}^-$ of dS_3 corresponding to the causal past of a timelike observer, whose holographic dual lives on a plane rather than a sphere. $\mathcal{O}^-$ can be foliated by asymptotically flat spacelike slices. Time evolution along these slices is generated by $L_0 + \bar{L}_0$, and is dual to scale transformations in the boundary CFT_2.

Keywords: AdS-CFT Correspondance, Cosmology of Theories beyond the SM.

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Contents

1. Introduction

The macroscopic entropy-area law [1, 2]

$$S = \frac{A}{4G}\,. \tag{1.1}$$

relates thermodynamic entropy to the area of an event horizon. A striking feature of this law is its universal applicability, including all varieties of black holes as well as de Sitter [3] and Rindler spacetimes. Understanding the microscopic origin of (1.1) is undoubtedly a key step towards understanding the fundamental nature of spacetime and quantum mechanics. Some progress has recently been made in deriving (1.1) for certain black holes in string theory [4]. This has led to a variety of insights culminating in the AdS/CFT correspondence [5]. However the situation remains unsatisfactory in that these recent developments do not fully explain the universality of (1.1).

In particular one would like to derive the entropy and thermodynamic properties of de Sitter space. This has taken on added significance with the emerging possibility that the real universe resembles de Sitter space [6]. Recent discussions of de Sitter thermodynamics include [7]–[20]. An obvious approach, successfuly employed in the black hole case, would be to begin by embedding de Sitter space as a solution of string theory, and then exploit various string dualities to obtain a microscopic description. Unfortunately persistent efforts by many (mostly unpublished!) have so far failed even to find a fully satisfactory de Sitter solution of string theory. Hopefully this situation will change in the not-too-distant future.

Meanwhile, string theory may not be the only route to at least a partial understanding of de Sitter space. Recall that the dual relation between AdS_3 and CFT_2 was discovered by Brown and Henneaux [21] from a general analysis of the asymptotic symmetries of anti-de Sitter space, and the central charge of the CFT_2 was computed. Later on black hole entropy was derived [22] using this central charge and Cardy's formula. In principle this required no input from string theory. Of course the arguments of [22] would have been less convincing without the concrete examples supplied by string theory.

In the absence of a stringy example of de Sitter space, in this paper we will sketch the parallel steps, beginning with an analysis of the asymptotic symmetries along the lines of [21], toward an understanding of de Sitter space. The endpoint will be a holographic duality relating quantum gravity on de Sitter space to a euclidean CFT on a sphere of one lower dimension. Our steps will be guided by the analogy to the AdS/CFT correspondence. We will see many similarities but also important differences between the AdS/CFT and proposed dS/CFT correspondences.

One of the first issues that must be faced in discussions of quantum de Sitter is the spacetime region under consideration. An initial reaction might be to consider the entire spacetime, which contains two boundaries $\mathcal{I}^{\pm}$ which are past and future spheres. This may ultimately be the correct view, but it is problematic for several reasons. The first is that a single immortal observer in de Sitter space can see at most half of the space. So a description of the entire space goes beyond what can be physically measured. Trying to describe the entirety of de Sitter space is like trying to describe the inside and outside of a black hole at the same time, and may lead to trouble. A second problem recently stressed in [23] is that, if enough matter is present, a space which is asymptotically de Sitter at $\mathcal{I}^-$ may collapse at finite time, and there will be no future $\mathcal{I}^+$ de Sitter region at all. Even when collapse does not occur, the presence of matter alters the causal structure [24]. This obscures the relevance of the global de Sitter geometry.

A smaller region[1], denoted $\mathcal{O}^-$ herein, is the region which can be seen be a single timelike observer in de Sitter space. It includes the planar past asymptotic region $\hat{\mathcal{I}}^-$, which is $\mathcal{I}^-$ minus a point, but not $\mathcal{I}^+$. Discussion of the quantum physics of $\mathcal{O}^-$ does not include unobservable regions and does not presume the existence of $\mathcal{I}^+$.

[1]References [7, 11, 14] advocate an even smaller causal region corresponding to the interior of both the past and future horizons of a timelike observer. This even smaller region excludes both $\mathcal{I}^-$ and $\mathcal{I}^+$.

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In this paper we will consider the holographic duals for both this region and the full space. One of our conclusions will be that in both cases the dual is a single euclidean CFT, despite the two boundaries of the full space, so from the dual perspective the two cases are not as different as they might seem.

We begin in section 2 with a discussion of the asymptotic symmetries of dS_3. It is shown that conformal diffeomorphisms of the spacelike surfaces can be compensated for by shifts in the time coordinate in such a way that the asymptotic form of the metric on $\hat{\mathcal{I}}^-$ is unchanged. This is similar to the AdS_3 case [21] except that time and radial coordinates are exchanged. Hence the asymptotic symmetry group of dS_3 is the euclidean conformal group in two dimensions. The global $SL(2,\mathbb{C})$ subgroup is the dS_3 isometry group. In section 3 we introduce the Brown-York stress tensor [25] for the boundary of dS_3. In section 4 we determine the central charge of the CFT (following [26]) from the anomalous variation of this boundary stress tensor. We find $c = \frac{3\ell}{2G}$, with ℓ the de Sitter radius and G Newton's constant. In section 5 we study correlators for massive scalar fields with points on $\hat{\mathcal{I}}^-$. It is found that they have the right form to be dual to CFT correlators of conformal fields on the plane, with conformal weights determined from the scalar mass. For $m^2\ell^2 > 1$, the conformal weights become complex. This means that the boundary CFT is not unitary if there are stable scalars with masses above this bound.

In section 6 we turn to global dS_3 which has two asymptotic S^2 regions. We first show that scalar correlators with points only on $\mathcal{I}^-$ are dual to CFT correlators on the sphere. We then consider the case with one point on $\mathcal{I}^-$ and one point on $\mathcal{I}^+$. These have singularities when the point on $\mathcal{I}^+$ is antipodal to the point on $\mathcal{I}^-$. This is because a light ray beginning on the sphere at $\mathcal{I}^-$ reaches its antipode at $\mathcal{I}^+$, and so antipodal points are connected by null geodesics. This causal connection relating points on $\mathcal{I}^-$ to those on $\mathcal{I}^+$ breaks the two copies of the conformal group (one each for $\mathcal{I}^\pm$) down to a single copy. After inverting the argument of the boundary field on $\mathcal{I}^+$, correlators with one point on $\mathcal{I}^+$ and one point on $\mathcal{I}^-$ have the same form as those with both points on $\mathcal{I}^-$ (or both on $\mathcal{I}^+$). Hence we propose that the dual CFT lives on a single euclidean sphere, rather than two spheres as naively suggested by the nature of the boundary of global dS_3.[2]

In section 7 we return to the region $\mathcal{O}^-$ of dS_3. This region can be foliated by asymptotically flat spacelike slices. Quantum states can be defined on these slices and at $\hat{\mathcal{I}}^-$ form a representation of the conformal group. Time evolution is generated by $L_0 + \bar{L}_0$. In the dual CFT this is scale transformations. This is the de Sitter analog of the scale-radius duality in AdS/CFT, and may have interesting implications for cosmology. We close in section 8 with a brief discussion of generalizations to higher than three dimensions. An appendix includes some details of dS_3 geometry and Green functions.

[2]Since according to this proposal, the boundary CFT lives on a sphere in any case, the question of whether or not there is an $\mathcal{I}^+$ at all might then be rephrased in terms of properties of the CFT.

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It will be evident to the reader that our understanding of the proposed dS/CFT correspondence is incomplete. A complete understanding will ultimately require an example of de Sitter quantum gravity. We hope that the present work will help guide us in what to look for.

The closest things we have at present to examples are Hull's spacelike D-branes [8, 13] and Chern-Simon de Sitter gravity [27, 28]. Hull performs timelike T-duality to turn for example $AdS_5 \times S_5$ to into $dS_5 \times H_5$, where H_5 is the hyperbolic 5-plane, and argues that the dual is a spacelike 3-brane. As discussed in [8, 13], this example is pathological because there are fields with the wrong sign kinetic term. Nevertheless it may be an instructive example for some purposes. Alternately, pure gravity in de Sitter space can be written as an $SL(2,\mathbb{C})$ Chern-Simon gauge theory, which is holographically dual to a reduced[3] $SL(2,\mathbb{C})$ WZW model on the boundary. This latter theory (before reduction) was studied extensively in [30] as the complexification of SU(2), but its status remains unclear. Both of these examples deserve further exploration.

The notion that quantum gravity in de Sitter space may have a euclidean holographic dual, possibly related to $\mathcal{I}^-$ and/or $\mathcal{I}^+$, has arisen in a number of places, including [8, 12, 17, 20].

2. Asymptotic symmetries of dS_3

The region $\mathcal{O}^-$ comprising the causal past of a timelike observer in de Sitter space is illustrated in figure 1.

The metric for a planar slicing of $\mathcal{O}^-$ is given by

$$\frac{ds^2}{\ell^2} = e^{-2t} dz d\bar{z} - dt^2 \,. \tag{2.1}$$

We use $\hat{\mathcal{I}}^-$ to denote the plane which is the past infinity of $\mathcal{O}^-$. An asymptotically past de Sitter geometry is one for which the metric behaves for $t \to -\infty$ as

$$\begin{aligned} g_{z\bar{z}} &= \frac{e^{-2t}}{2} + \mathcal{O}(1) \,, \\ g_{tt} &= -1 + \mathcal{O}(e^{2t}) \,, \\ g_{zz} &= \mathcal{O}(1) \,, \\ g_{zt} &= \mathcal{O}(e^{2t}) \,. \end{aligned} \tag{2.2}$$

These boundary conditions are an analytic continuation of the AdS_3 boundary conditions of Brown and Henneaux [21]. The asymptotic symmetries of dS_3 are diffeomorphisms which preserve (2.2). Consider the vector fields

$$\zeta = U\partial_z + \frac{1}{2} e^{2t} U'' \partial_{\bar{z}} + \frac{1}{2} U' \partial_t \,, \tag{2.3}$$

[3] As in the AdS case [29], we expect this becomes Liouville theory after imposing the appropriate boundary conditions. Some discussion can be found in [9].

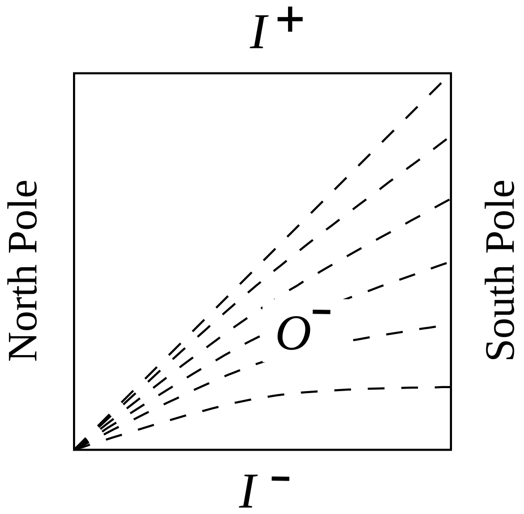

Figure 1: Penrose diagram for dS_3. Every point in the interior of the diagram is an S^1. A horizontal line is an S^2, with the left (right) vertical boundary being the north (south) pole. $\mathcal{O}^-$ is the region below the diagonal, and comprises the causal past of an observer at the south pole. The dashed lines are non-compact surfaces of constant t.

where $U = U(z)$ is holomorphic and the prime denotes differentiation. In order to obtain a real vector field one must add the complex conjugate, however for notational simplicity we suppress this addition in (2.3) and all subsequent formula. In general the metric transforms under a diffeomorphism as the Lie derivative

$$\delta_\zeta g_{mn} = -\mathcal{L}_\zeta g_{mn} \,. \tag{2.4}$$

For ζ parametrized by U as in (2.3), (2.4) becomes

$$\begin{aligned} \delta_U g_{zz} &= -\frac{\ell^2}{2} U''' \,, \\ \delta_U g_{z\bar{z}} &= \delta_U g_{zt} = \delta_U g_{tt} = 0 \,. \end{aligned} \tag{2.5}$$

The change (2.5) in the metric satisfies (2.2) and so (2.3) generates an asymptotic symmetry of de Sitter space on $\hat{\mathcal{I}}^-$. A special case of (2.3) is

$$U = \alpha + \beta z + \gamma z^2 \,, \tag{2.6}$$

where α, β, γ are complex constants. In this case U''' vanishes, and the metric is therefore invariant. These transformations generate the $SL(2, \mathbb{C})$ global isometries of 2+1 de Sitter.

In conclusion the asymptotic symmetry group is the conformal group of the complex plane, and the isometry group is the $SL(2, \mathbb{C})$ subgroup of the asymptotic symmetry group.

3. The boundary stress tensor

Brown and York [25] have given a general prescription for defining a stress tensor associated to a boundary of a spacetime. Our treatment parallels the discussion of AdS_3 in [26]. We will be interested in the boundary $t \to -\infty$ at $\hat{\mathcal{I}}^-$. In the case at hand the stress tensor is[4]

$$T^{\mu\nu} = -\frac{4\pi}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}}, \tag{3.1}$$

where γ is the induced metric on the boundary. The action is

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \left(R - \frac{2}{\ell^2} \right) + \frac{1}{8\pi G} \int_{\partial \mathcal{M}} \sqrt{\gamma} K + \frac{1}{8\pi G \ell} \int_{\partial \mathcal{M}} \sqrt{\gamma} + S_{\text{matter}}, \tag{3.2}$$

K here is the trace of the extrinsic curvature defined by $K_{\mu\nu} = -\nabla_{(\mu} n_{\nu)} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$ with n^μ the outward pointing unit normal. The second integral in (3.2) is the usual gravitational surface term. The third integral is a surface counterterm required for finiteness of T at an asymptotic boundary, and uniquely fixed by locality and general covariance [26]. The matter action is assumed not to be relevant near the boundary and will henceforth be suppressed.

The sign in the definition (3.1) leads to a positive mass[5] for Schwarzschild-de Sitter [32]. It would be interesting to see if (3.1) reproduces the canonical operator product expansion for a 2D CFT stress tensor.

Using (3.2) to evaluate (3.1) we learn that, for solutions of the bulk equations of motion,

$$T^{\mu\nu} = \frac{1}{4G} \left[K^{\mu\nu} - (K + \frac{1}{\ell}) \gamma^{\mu\nu} \right]. \tag{3.3}$$

This vanishes for de Sitter space on $\hat{\mathcal{I}}^-$ in the coordinates (2.1). For more general asymptotically dS_3 spacetimes (obeying (2.2)) (3.3) implies

$$T_{zz} = \frac{1}{4G} \left[K_{zz} + \frac{1}{\ell} \gamma_{zz} \right]. \tag{3.4}$$

4. The central charge

A central charge can be associated to dS_3 by analyzing the behavior of the stress tensor on $\mathcal{I}^-$. We follow related discussions for AdS_3 [21, 26] (which in turn followed the earlier work [33]). Under the conformal transformations (2.3), one finds

$$\delta_U T_{zz} = -\frac{\ell}{8G} U'''. \tag{4.1}$$

[4] Our conventions in this section are those of [25, 26] except for a factor of -2π in this equation, in order to conform with the more standard conventions [31] for 2D CFT stress tensor.

[5] More precisely, a positive value of the AD mass $L_0 + \bar{L}_0$, as defined below in equation 7.4.

This transformation identifies the central charge as,[6]

$$c = \frac{3\ell}{2G}\,. \tag{4.2}$$

It is presumably also possible to derive c by an alternate method [34, 33, 26] which relates it to the trace anomaly by using spherical rather than planar spatial sections.

5. The plane and $\hat{\mathcal{I}}^-$ correlators

In the previous sections we have seen that the conformal group of euclidean R^2 has an action on $\hat{\mathcal{I}}^-$. One therefore expects that appropriately rescaled gravity correlators restricted to $\hat{\mathcal{I}}^-$ will be those of a euclidean 2D conformal field theory. In this section we verify this expectation for the case of a massive scalar.

Consider a scalar field of mass m with wave equation

$$m^2\ell^2\phi = \ell^2\nabla^2\phi = -\partial_t^2\phi + 2\partial_t\phi + 4e^{2t}\partial_z\partial_{\bar{z}}\phi\,. \tag{5.1}$$

Near $\hat{\mathcal{I}}^-$ the last term in (5.1) is negligible and solutions behave as

$$\phi \sim e^{h_\pm t}\,, \qquad t \to -\infty\,, \tag{5.2}$$

where

$$h_\pm = 1 \pm \sqrt{1 - m^2\ell^2}\,. \tag{5.3}$$

We first consider the case $0 < m^2\ell^2 < 1$ so that $h_\pm$ are real and $h_- < 1 < h_+$. As a boundary condition on $\hat{\mathcal{I}}^-$ we demand

$$\lim_{t\to-\infty} \phi(z,\bar{z},t) = e^{h_- t}\phi_-(z,\bar{z})\,, \tag{5.4}$$

with corrections suppressed by at least one power of e^{2t}. The dS_3 /CFT_2 correspondence proposes, in direct analogy with the AdS/CFT correspondence, that ϕ_- is dual to an operator $\mathcal{O}_\phi$ of dimension h_+ in the boundary CFT. The two point correlator of $\mathcal{O}_\phi$ is (up to normalization) the quadratic coefficient of ϕ_- in the expression[7]

$$\lim_{t\to-\infty}\int_{\hat{\mathcal{I}}^-} d^2z d^2v [e^{-2(t+t')}\phi(t,z,\bar{z})\overleftrightarrow{\partial}_t G(t,z,\bar{z};t',v,\bar{v})\overleftrightarrow{\partial}_{t'}\phi(t',v,\bar{v})]_{t=t'}\,. \tag{5.5}$$

[6] This equation was arrived at from a different perspective in [9] and a related equation appears in [11].

[7] This is equivalent to the usual expression used in AdS/CFT (except of course with a different boundary and G), as can be readily seen from the formula for the bulk Green function G in terms of the bulk-to-boundary Green function. A clear discussion can be found in [35] . We have avoided use of the bulk-to-boundary Green function because in the de Sitter case we need to keep track of both terms in (5.6).

G here is the de Sitter invariant Green function given in the appendix. Near $\hat{\mathcal{I}}^-$ it reduces to

$$\lim_{t,t'\to-\infty} G(t,z,\bar{z};t',v,\bar{v}) = \frac{c_+ e^{h_+(t+t')}}{|z-v|^{2h_+}} + \frac{c_- e^{h_-(t+t')}}{|z-v|^{2h_-}}, \tag{5.6}$$

where $c_\pm$ are constants. Inserting (5.4) and (5.6), (5.5) is proportional to[8]

$$\int_{\hat{\mathcal{I}}^-} d^2z d^2v \phi_-(z,\bar{z})|z-v|^{-2h_+}\phi_-(v,\bar{v}). \tag{5.7}$$

We conclude that the dual operator $\mathcal{O}_\phi$ obeys

$$\langle \mathcal{O}_\phi(z,\bar{z})\mathcal{O}_\phi(v,\bar{v})\rangle = \frac{\text{const.}}{|z-v|^{2h_+}}, \tag{5.8}$$

as is appropriate for an operator of dimension h_+.

It should be noted that the boundary conditions (5.4) are not the most general. There are also solutions with the subleading behavior $\phi_+(z,\bar{z})e^{h_+t}$ at $\hat{\mathcal{I}}^-$. Including these would lead to an additional term in (5.7) proportional to

$$\int_{\hat{\mathcal{I}}^-} d^2z d^2v \phi_+(z,\bar{z})|z-v|^{-2h_-}\phi_+(v,\bar{v}), \tag{5.9}$$

which might be associated with an operator of dimension h_-. In the next section we will find that this extra boundary condition can imposed on the second boundary at $\mathcal{I}^+$, which is not within the coordinate patch $\mathcal{O}^-$ covered in the planar coordinates (2.1). Alternately this second set of independent fields might be eliminated by imposing a suitable boundary condition at the future horizon $t\to\infty$, which would lead to a different Green function.[9]

What happens if $m^2\ell^2 > 1$? In that case the conformal weight h_- is no longer real. $h_\pm$ are complex conjugates with real part equal to unity. Of course the appearance of imaginary conformal weights suggests that the dual CFT is not unitary. This might mean that consistent theories of de Sitter quantum gravity have no stable scalars with $m^2\ell^2 > 1$. On the other hand we know of no obvious reason that the dual CFT needs to be unitary.

The preceding discussion is reminiscent of Liouville theory and indeed suggests that the boundary CFT has a Liouville-like form. In some discussions of Liouville theory operators with complex dimensions are encountered. Further there are various kinds of operators (called macroscopic and microscopic in [37]) which may or may not be allowed depending on the context.

[8]Similar holographic expressions were derived for general coset spaces in [36] , but the case of de Sitter space was not explicitly considered.

[9]I thank J. Maldacena for this suggestion.

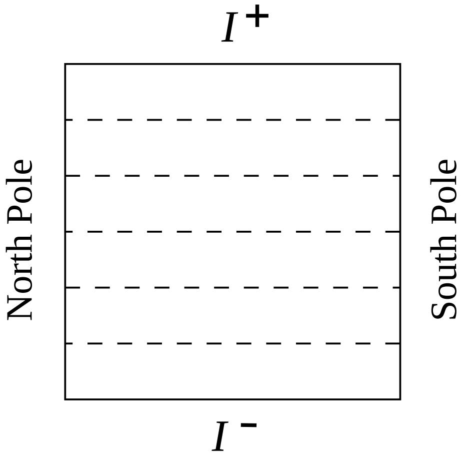

Figure 2: Lines of constant τ in global spherical coordinates are the spacelike two-spheres indicated by dashed lines.

6. The sphere and $\mathcal{I}^{\pm}$ correlators

An alternate form of the dS_3 metric is

$$\frac{ds^2}{\ell^2} = -d\tau^2 + 4\cosh^2\tau \frac{dwd\bar{w}}{(1+w\bar{w})^2} \tag{6.1}$$

$w = \tan\frac{\theta}{2}e^{i\phi}$ here is a complex coordinate on the round sphere. This metric describes dS_3 as a contracting/expanding two-sphere. These coordinates cover the entire space which has future and past S^2 boundaries $\mathcal{I}^{\pm}$, as depicted in figure 2. In general we can consider correlators with points on either or both of the boundaries.

We begin with the two-point correlator with both points on $\mathcal{I}^-$. We wish to compute the spherical analog of (5.5), which is

$$\lim_{\tau\to-\infty}\int_{\mathcal{I}^-} d^2wd^2v\sqrt{h(w)h(v)}\times \\ \times\left[e^{-2(\tau+\tau')}\phi(\tau,w,\bar{w})\overleftrightarrow{\partial}_{\tau}G(\tau,w,\bar{w};\tau',v,\bar{v})\overleftrightarrow{\partial}_{\tau'}\phi(\tau',v,\bar{v})\right]_{\tau=\tau'}, \tag{6.2}$$

where $h(w) = 2(1+w\bar{w})^{-2}$ is the measure on the sphere. Near $\mathcal{I}^-$, ϕ can be decomposed as

$$\lim_{\tau\to-\infty}\phi(\tau,w,\bar{w}) = \phi_+^{\rm in}(w,\bar{w})e^{h_+\tau} + \phi_-^{\rm in}(w,\bar{w})e^{h_-\tau}. \tag{6.3}$$

The superscripts "in" ("out") are used to denote quantities on $\mathcal{I}^-$ ($\mathcal{I}^+$). As seen in the appendix, the propagator behaves as

$$\lim_{\tau,\tau'\to-\infty}G(\tau,w,\bar{w};\tau',v,\bar{v}) = c_+e^{h_+(\tau+\tau')}\frac{(1+w\bar{w})^{h_+}(1+v\bar{v})^{h_+}}{|w-v|^{2h_+}} + \\ +c_-e^{h_-(\tau+\tau')}\frac{(1+w\bar{w})^{h_-}(1+v\bar{v})^{h_-}}{|w-v|^{2h_-}}. \tag{6.4}$$

(6.2) is then proportional to

$$\int_{\mathcal{I}^-} d^2v d^2w \sqrt{h(v)h(w)}(c_+\phi_-^{\text{in}}(w,\bar{w})\Delta_{h_+}(w,\bar{w};v,\bar{v})\phi_-^{\text{in}}(v,\bar{v}) + \\ + c_-\phi_+^{\text{in}}(w,\bar{w})\Delta_{h_-}(w,\bar{w};v,\bar{v})\phi_+^{\text{in}}(v,\bar{v}))\,. \tag{6.5}$$

$\Delta_{h_\pm}$ here is the two point function for a conformal field of dimension $h_\pm$ on the sphere:

$$\Delta_{h_\pm} = \left[\frac{(1+w\bar{w})(1+v\bar{v})}{|w-v|^2}\right]^{h_\pm}\,, \tag{6.6}$$

including the normalization factor from the Weyl anomaly on a curved geometry. Hence we see that, as in the planar case, the two-point scalar correlators can be identified with correlators of conformal fields of dimension $h_\pm$, except that now they are on the sphere rather than the plane. A similar expression holds for the boundary at $\mathcal{I}^+$.

Life becomes more interesting when we put one point on $\mathcal{I}^-$ and one on $\mathcal{I}^+$. Then we must compute

$$\lim_{\tau\to-\infty}\int_{\mathcal{I}^-} d^2w \int_{\mathcal{I}^+} d^2v \sqrt{h(w)h(v)}\,\times \\ \times\,[e^{2(\tau-\tau')}\phi(\tau,w,\bar{w})\overleftrightarrow{\partial}_\tau G(\tau,w,\bar{w};\tau',v,\bar{v})\overleftrightarrow{\partial}_{\tau'}\phi(\tau',v,\bar{v})]_{\tau=-\tau'}\,. \tag{6.7}$$

For this case, as shown in the appendix, the relevant limit of the Green function is

$$\lim_{\tau\to-\infty,\tau'\to+\infty} G(\tau,w,\bar{w};\tau',v,\bar{v}) = c_+\cos(\pi h_+)e^{h_+(\tau-\tau')}\Delta_{h_+}\left(w,\bar{w};-\frac{1}{\bar{v}},-\frac{1}{v}\right) + \\ +(h_+\leftrightarrow h_-)\,. \tag{6.8}$$

It is convenient to define the inverted boundary field at $\mathcal{I}^+$

$$\tilde{\phi}_+^{\text{out}}(v,\bar{v}) = \phi_+^{\text{out}}\left(-\frac{1}{\bar{v}},-\frac{1}{v}\right)\,. \tag{6.9}$$

Then (6.7) is proportional to

$$\int_{S^2} d^2w d^2v \sqrt{h(w)h(v)}(c_+\cos(\pi h_+)\phi_-^{\text{in}}(w,\bar{w})\Delta_{h_+}(w,\bar{w};v,\bar{v})\tilde{\phi}_-^{\text{out}}(v,\bar{v}) + \\ + c_-\cos(\pi h_-)\phi_+^{\text{in}}(w,\bar{w})\Delta_{h_-}(w,\bar{w};v,\bar{v})\tilde{\phi}_+^{\text{out}}(v,\bar{v}))\,. \tag{6.10}$$

In particular we see that ϕ^{in} and $\tilde{\phi}^{\text{out}}$ have a non-trivial two point function despite the fact that they live on widely separated boundary components.

(6.9) and (6.10) can be interpreted as follows. Bulk gravity correlators with all points on $\mathcal{I}^-$ are CFT correlators on the sphere. Inserting additional gravity operators on $\mathcal{I}^+$ corresponds to inserting the dual CFT operator at the antipodal point of the sphere. The reason for this inversion of insertions at $\mathcal{I}^+$ is simple. A singularity of a correlator between a point on $\mathcal{I}^-$ and one on $\mathcal{I}^+$ can occur only if the two points are connected by a null geodesic. A light ray beginning on the sphere at $\mathcal{I}^-$ reaches the antipodal point of the sphere at $\mathcal{I}^+$, as can be easily seen from figure 2. Therefore there is an inversion in the map from the sphere at $\mathcal{I}^+$ to the one at $\mathcal{I}^-$.[10]

This causal connection between points at $\mathcal{I}^+$ and $\mathcal{I}^-$ has important consequences for the symmetry group. Naively one might have expected two copies of the conformal group, one for $\mathcal{I}^+$ and one for $\mathcal{I}^-$, and correspondingly two separate CFTs. However the Green functions know about the causal connection between points and therefore transform simply only under a subgroup of the two conformal groups. The result is a single conformal group and a single CFT on a single sphere.

We note that only two of the four boundary fields $\phi_\pm^{\rm in}$, $\phi_\pm^{\rm out}$ are independent, the remaining two being determined by the equation of motion (at the semiclassical level). These relations are the much-studied [38] Bogolubov transformations relating $\mathcal{I}^+$ modes to $\mathcal{I}^-$ modes. Therefore there are at most two independent boundary operators.

One might alternately have employed one of the other Green functions in (6.2) or (6.7) , such as the Feynman Green function, or a Green function associated with one of the other de Sitter invariant vacua. Such modifications remain to be explored.

7. Quantum states and Virasoro generators

A quantum state in the patch $\mathcal{O}^-$ can be characterized by its wave function Ψ on the plane $\hat{\mathcal{I}}^-$. Ψ is a functional on the space of asymptotically euclidean (on the $\hat{\mathcal{I}}^-$ plane) two-geometries γ. Since the complex diffeomorphisms (2.3) map this space to itself, the states Ψ form a representation of the conformal group. That is, they are states in a conformal field theory.

The states Ψ are most naturally described in a radial quantization of the dual CFT as wave functions on the S^1 boundary of the $\hat{\mathcal{I}}^-$. Radial evolution is generated by $L_0 + \bar{L}_0$. In the bulk description this operator generates Killing flow along $z\partial_z + \bar{z}\partial_{\bar{z}} + \partial_t$, as depicted in figure 3.[11] So it generates time evolution along the planar spacelike slices in (2.1), accompanied by a dilation. At large radius the norm of the dilation grows and the Killing vector becomes spacelike.[12] The eigenvalue of $L_0 + \bar{L}_0$

[10]When back reaction is included, the geometry is perturbed, and light rays tend to pass the antipodal point [24]. Hence this identification may be deformed in perturbation theory. I thank R. Bousso for discussions of this point.

[11]We note the absence of a factor of i here. In contrast on the 2D Minkowski cylinder $L_0 + \bar{L}_0$ generates $i\partial_t$. This may be related to the thermal nature of de Sitter space.

[12]An analogy would be the operator $H + J$, where H is an ordinary Hamiltonian and J a rotation operator. At large radius the motion generated by this operator is spacelike, nevertheless it generates evolution along spacelike slices.

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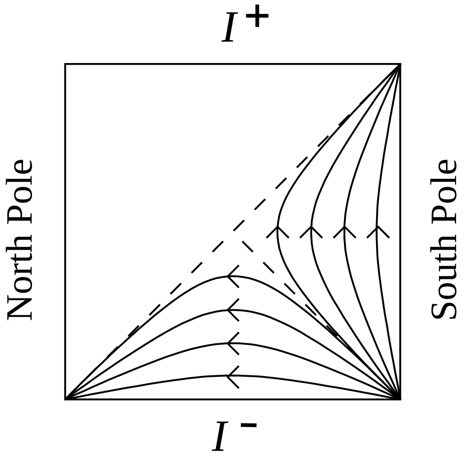

Figure 3: The arrows indicate the direction of the Killing flow generated by $L_0 + \bar{L}_0$ within $\mathcal{O}^-$. Note that it is timelike along the worldline of an observer at the south pole but becomes spacelike on $\hat{\mathcal{I}}^-$.

is (up to a constant) a conserved charge known as the AD mass [39].

As usual, the generators $H(\zeta)$ of any of the diffeomorphisms $\zeta(U)$ can be written as a surface integral at infinity (in this case the circle $z\bar{z} \to \infty$) after gauge fixing, imposing the constraints and constructing the Dirac brackets. The full expression, as given in [21], is

$$H(\zeta) = \frac{1}{16\pi G}\int dS_\mu \{[\sqrt{\gamma}(\gamma^{\mu\lambda}\gamma^{\nu\rho} - \gamma^{\mu\nu}\gamma^{\lambda\rho})(\zeta^t \delta\gamma_{\lambda\rho;\nu} - \zeta^t_{,k}\delta\gamma_{\lambda\rho})] + \\ + 2\zeta_\nu \delta\pi^{\mu\nu} + (2\zeta^\lambda \pi^{\rho\mu} - \zeta^\mu \pi^{\lambda\rho})\delta\gamma_{\lambda\rho}\}\,. \tag{7.1}$$

In this expression, $\pi^{\mu\nu} = \sqrt{\gamma}(K^{\mu\nu} - K\gamma^{\mu\nu})$ is $16\pi G$ times the momentum conjugate to $\gamma_{\mu\nu}$, and the prefix δ denotes the deviation of the metric and momentum from their fiducial dS_3 values $\gamma_{0z\bar{z}} = \frac{\ell^2}{2}e^{-2t}$ and $\pi_0^{z\bar{z}} = 1$. Imposing the boundary conditions (2.2) and using expression (2.3) for ζ we find many terms (including all those in the square parenthesis) vanish as one approaches $\hat{\mathcal{I}}^-$. (7.1) simplifies to

$$H(\zeta) = -\frac{i}{8\pi G\ell}\int dz(\zeta^z \pi^{z\bar{z}}\gamma_{zz} + \zeta^z \gamma_{z\bar{z}}\pi^{\bar{z}\bar{z}})\,. \tag{7.2}$$

In terms of the boundary stress tensor given in (3.1), this becomes

$$H(\zeta) = \frac{1}{2\pi i}\int dz T_{zz}\zeta^z\,. \tag{7.3}$$

Defining $\zeta_n = \zeta(z^{n+1})$, we have

$$L_n \equiv H(\zeta_n) = \frac{1}{2\pi i}\int dz T_{zz} z^{n+1}\,. \tag{7.4}$$

Expression (7.4) can also be more directly derived in the formalism of [25] , where for every boundary symmetry ζ^ν, there is an associated conserved current $T_{\mu\nu}\zeta^\nu$. The associated charge is then just the integral of the normal component of the current around the contour, which is precisely expression (7.4).

It is tempting to try to compute the de Sitter entropy by applying the Cardy formula to these states. This will be explored in [32].

8. dS_D/CFT_{D-1} correspondence

The dS_3/CFT_2 correspondence discussed in the preceding sections has an obvious generalization to higher dimensions which we briefly mention in this section. It states that bulk quantum gravity on dS_D is holographically dual to a euclidean conformal field theory on S^{D-1}. The planar metric for dS_D is

$$\frac{ds^2}{\ell^2} = -dt^2 + e^{-2t} d\vec{x} \cdot d\vec{x} \,, \tag{8.1}$$

while the spherical metric is

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2 \tau d\Omega^2_{D-1} \,, \tag{8.2}$$

with $d\Omega^2_{D-1}$ the unit metric on S^{D-1}. It can be seen from (8.2) that in de Sitter space of any dimension that a light ray on $\mathcal{I}^-$ reaches the antipodal point of the sphere at $\mathcal{I}^+$. Therefore the boundary CFT should always involve a single sphere, but the arguments of bulk correlators on $\mathcal{I}^+$ should map to antipodal points on the sphere, relative to those from $\mathcal{I}^-$. One also finds from the asymptotic behavior of the wave equation that equation (5.3) for the conformal weights is generalized to

$$h_\pm = \frac{1}{2}((D-1) \pm \sqrt{(D-1)^2 - 4m^2\ell^2}) \,. \tag{8.3}$$

Again we see complex conformal weights for sufficiently massive states.

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A. dS_3 Geometry

dS_3 is described by the hyperboloid

$$X^2 + Y^2 + Z^2 - T^2 = \ell^2 \tag{A.1}$$

in 3+1 Minkowski space. The planar coordinates (z,t) are defined by

$$\begin{aligned} t &= -\ln\frac{Z-T}{\ell}\,, \\ z &= \frac{X+iY}{Z-T}\,, \\ Z-T &= \ell e^{-t}\,, \\ Z+T &= \ell e^{t} - \ell z\bar{z}e^{-t}\,, \\ X+iY &= \ell z e^{-t}\,. \end{aligned} \tag{A.2}$$

These lead to the metric (2.1)

$$\frac{ds^2}{\ell^2} = e^{-2t}dzd\bar{z} - dt^2\,. \tag{A.3}$$

The spherical coordinates (τ, w) in (6.1) are defined by

$$\begin{aligned} T &= \ell\sinh\tau\,, \\ Z &= \ell\frac{1-w\bar{w}}{1+w\bar{w}}\cosh\tau\,, \\ X+iY &= \frac{2\ell w\cosh\tau}{1+w\bar{w}}\,. \end{aligned} \tag{A.4}$$

These give the metric

$$\frac{ds^2}{\ell^2} = -d\tau^2 + 4\cosh^2\tau\frac{dwd\bar{w}}{(1+w\bar{w})^2}\,, \tag{A.5}$$

where $w = \tan\frac{\theta}{2}e^{i\phi}$ is a complex coordinate on the round sphere. The relation between the spherical coordinates and the planar coordinates is

$$\begin{aligned} z &= \frac{w(1+e^{2\tau})}{1-w\bar{w}e^{2\tau}}\,, \\ t &= \tau - \ln\left[\frac{1-w\bar{w}e^{2\tau}}{1+w\bar{w}}\right]\,. \end{aligned} \tag{A.6}$$

The geodesic distance $d(X,X')$ between two points X and X' has a simple expression in terms of the Minkowski coordinates. Define

$$\ell^2 P(X,X') = XX' + YY' + ZZ' - TT'\,. \tag{A.7}$$

Then

$$d = \ell\cos^{-1}P\,. \tag{A.8}$$

De Sitter invariance implies that the Hadamard two point function

$$G(X,X') = \text{const.}\langle 0|\{\phi(X),\phi(X')\}|0\rangle \tag{A.9}$$

in a de Sitter invariant vacuum state is a function only of d, or equivalently P. Hence away from singularities G obeys

$$(P^2-1)\partial_P^2 G + 3P\partial_P G + m^2\ell^2 G = 0\,. \tag{A.10}$$

This equation has two linearly independent solutions (related by $P \to -P$), corresponding to the existence of a one-parameter family of de Sitter invariant vacua. Among these only one has singularities only along the light cone $P = 1$. This solution is the hypergeometric function

$$G(P) = \mathrm{Re}F\left(h_+, h_-, \frac{3}{2}; \frac{1+P}{2}\right). \tag{A.11}$$

In planar coordinates one finds near $\hat{\mathcal{I}}^-$:

$$\lim_{t,t'\to-\infty} P(t,z,\bar{z};t',v,\bar{v}) = -\frac{1}{2}e^{-t-t'}|z-v|^2\,. \tag{A.12}$$

This diverges, so G can be evaluated with the aid of the transformation formula,

$$\begin{aligned} F\left(h_+, h_-, \frac{3}{2}; z\right) = &\frac{\Gamma(3/2)\,\Gamma(h_- - h_+)}{\Gamma(h_-)\Gamma(3/2-h_+)(-z)^{-h_+}} \times \\ &\times F\left(h_+, h_+ - \frac{1}{2}, h_+ + 1 - h_-; \frac{1}{z}\right) + (h_+ \leftrightarrow h_-) \end{aligned} \tag{A.13}$$

and $F(\alpha,\beta,\gamma;0) = 1$. One finds

$$\lim_{t,t'\to-\infty} G(t,z,\bar{z};t',v,\bar{v}) = \frac{4^{h_+}\Gamma(3/2)\Gamma(h_- - h_+)}{\Gamma(h_-)\Gamma(3/2-h_+)}\frac{e^{h_+(t+t')}}{|z-v|^{2h_+}} + (h_+ \leftrightarrow h_-)\,, \tag{A.14}$$

as given in (5.6).

In spherical coordinates one finds near $\mathcal{I}^-$

$$\lim_{\tau,\tau'\to-\infty} P(\tau,w,\bar{w};\tau',v,\bar{v}) = -\frac{e^{-\tau-\tau'}|w-v|^2}{2(1+w\bar{w})(1+v\bar{v})}\,. \tag{A.15}$$

Using (A.13) then leads to equation (6.4) for G on $\mathcal{I}^-$. We also need G for the case that τ' approaches $\mathcal{I}^+$ while τ approaches $\mathcal{I}^-$. This can be deduced from the fact that inverting one of the arguments of P (i.e. $X \to -X$) simply changes it sign. Hence

$$P(\tau,w,\bar{w};\tau',v,\bar{v}) = -P\left(\tau,w,\bar{w};-\tau',-\frac{1}{\bar{v}},-\frac{1}{v}\right). \tag{A.16}$$

Hence $P \to +\infty$ for one point on $\mathcal{I}^-$ and one on $\mathcal{I}^+$. F has a singularity at $P = 1$ and a branch cut extending from from $P = 1$ to infinity. Using the fact that G is the real part of F then yields (6.8) .

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